Scaling properties in off-equilibrium dynamical processes

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In this paper, we analyze the consequences of scaling hypotheses on dynamic functions, such as two-time correlations $C(t,t')$. We show, under general conditions, that $C(t,t')$ must obey the scaling behavior $C(t,t')=\phi_1(t)^{f(\beta)}\mathcal{S}(\beta)$, where the scaling variable is $\beta=\beta(\phi_1(t')/\phi_1(t))$ and $\phi_1(t'),\phi_1(t)$ are two undetermined functions. The presence of a nonconstant exponent $f(\beta)$ signals the appearance of multiscaling properties in the dynamics. [S1063-651X(99)15003-7]

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I. INTRODUCTION

The introduction of scaling concepts to describe equilibrium and off-equilibrium dynamics in statistical mechanics was originally motivated by experimental and simulation data about, for instance, structure factor, pair correlation functions, and response functions. Actually, the study of several classes of materials with complex dynamical properties such as magnets, polymers, glasses, and several other thermal systems, and even nonthermal systems such as granular media, has shown the presence of some general scaling features [1–3]. In order to formulate a coherent scaling approach to the dynamics of systems out of equilibrium, in this paper we resort to a general scheme developed in 1971 [4]. This approach also reproduces as a particular case the multifractal and the multiscaling formalisms, which have been applied to a large variety of phenomena such as turbulence, random resistor networks, self-organized criticality, spinodal decomposition, and many more [5–13]. The general scaling formulation applied to systems out of equilibrium stems from the hypothesis of invariance of two-time functions, such as autocorrelation functions, under a general scaling transformation with the only requirement that the transformation obey group properties.

For definiteness let us consider a two-time correlation function, $C(t,t')$, which, for example, could be the density-density autocorrelation function in a supercooled liquid or the spin-spin correlation function in a magnetic system. We suppose that the system is prepared at time $t=0$ and is probed at two subsequent times t' and t. When the system is out of equilibrium, $C(t,t')$ generally depends on both t' and t. Whenever the relaxation characteristic time τ is very large or infinite, we may make the following asymptotic scaling ansatz valid for t and t' large but smaller than τ: by rescaling the system lengths by a factor l, if t and t' are opportunely rescaled, we may expect that the autocorrelation function scales as $l^{f(t,t')}$, where the exponent $f(t,t')$ is, in general, dependent on t and t' [14]. To be precise, we assume that the function $C(t,t')$ has the following general scaling property:

$$C(\tilde{t},\tilde{t}')=l^{-f(t,t')}C(t,t') \tag{1}$$

under a general time rescaling ''*nonmixing*'' transformation, which satisfies group rules, such as

$$\tilde{t}=F_1(t,l) \quad \tilde{t}'=F_2(t',l), \tag{2}$$

with the condition that $F_i(x,1)=x(i=1,2)$. The above transformations are nonmixing in the sense that $F_1(F_2)$ depends only on $t(t')$. The requirement that the transformation obey group properties imposes some constraints on the functions $C(t,t')$ and $f(t,t')$. Interestingly, under these assumptions, we find that $C(t,t')$ can be synthetically expressed in the following way:

$$C(t,t')=\phi_1(t)^{f(\beta)}\mathcal{S}(\beta), \tag{3}$$

where the scaling variable β has the following form:

$$\beta=\phi_2(t')/\phi_1(t). \tag{4}$$

Here the $\phi_i(i=1,2)$ are two unknown functions fixed by the transformations given in Eq. (2). Equation (3) in the particular case $f(\beta)=0$ was obtained in Ref. [15] using different arguments. Notice that whenever f is not a constant, a ''multiscaling'' dynamical behavior is found in the dynamics, an interesting issue to check in models as well as experiments and simulations.

II. GENERAL NONMIXING CASE

In what follows we give a demonstration of the principles summarized above. As shown in Ref. [4], the general transformations of Eq. (2) implies that there exist a couple of functions, $\phi_1(t)$ and $\phi_2(t')$, that under rescaling exhibit the following properties:

$$\phi_1(\tilde{t})=\phi_1(t)/l, \quad \phi_2(\tilde{t}')=\phi_2(t')/l. \tag{5}$$

These equations state that the ''true'' scaling variables are the ϕ_i's and that whenever the functions ϕ_i are *invertible*, Eqs. (2) can be expressed in the following way:

$$\tilde{t}\equiv F_1(t,l)=\phi_1^{-1}(\phi_1(t)/l), \quad \tilde{t}'\equiv F_2(t',l)=\phi_2^{-1}(\phi_2(t')/l), \tag{6}$$

where ϕ_i^{-1} is the inverse function of ϕ_i [i.e., $\phi_i^{-1}(\phi_i(x))=x$].

Let us now study how the above group properties influence the structure of the function $C(t,t')$. The group rules impose that if we scale t and t' by a factor l_1 and later by a factor l_2 this should be equivalent to rescaling them by a factor $l_1 l_2$. More formally, we can express this condition as

$$C[F_1(F_1(t,l_1),l_2),F_2(F_2(t',l_1),l_2)]=(l_1 l_2)^{-f(t,t')}C(t,t'). \tag{7}$$

Substituting Eqs. (1) and (2) in the above relation, one is led to a simple equation that states that

$$\frac{d}{dl}f(t(l),t'(l))=0, \tag{8}$$

where, by definition, $t(l)=F_1(t,l)$ and $t'(l)=F_2(t,l)$. By inserting Eq. (6) in Eq. (8), one finds that $f(\phi_1^{-1}(\phi_1/l),\phi_2^{-1}(\phi_2/l))=f(\phi_1^{-1}(\phi_1/1),\phi_2^{-1}(\phi_2/1))$, i.e., we have that $f(\phi_1^{-1}(\phi_1/l),\phi_2^{-1}(\phi_2/l))=f(t,t')$. Now, by taking $l=\phi_1$ we obtain that $f(t,t')=f(\phi_1^{-1}(1),\ \phi_2^{-1}(\phi_2/\phi_1))$, that is to say, $f(t,t')=f(\phi_2(t')/\phi_1(t))$.

Analogously, by inserting Eq. (6) in Eq. (1), and choosing $l=\phi_1$, we find the scaling form for $C(t,t')$ that we anticipated in Eq. (3) above. In such a way we also individuate the scaling function $\mathcal{S}$:$\mathcal{S}(x)=C(\phi_1^{-1}(1),\phi_2^{-1}(x))$.

Thus we proved that in the presence of scaling properties such as those written in Eqs. (1) and (2), the asymptotic functional form of the scaling of $C(t,t')$ is characterized by the asymptotic behavior of the ''true'' scaling variables ϕ_1 and ϕ_2, as written in the general result of Eq. (3).

For the sake of clarity we have dealt with a two-variable function, $C(t,t')$, but analogous properties may be proven for a many-variable function, $C(t_1,t_2,\ldots,t_n)$. In this case, if the generic variable undergoes a scale transformation $\tilde{t}_i=F_i(t_i,l)(i\in\{1,\ldots,n\})$, we have

$$C(t_1,t_2,\ldots,t_n)=\phi_1(t_1)^{f(\beta_2,\ldots,\beta_n)}\mathcal{S}(\beta_2,\ldots,\beta_n), \tag{9}$$

with $(i>1)\beta_i=\phi_i(t_i)/\phi_1(t_1)$. As before, $f(x_1,\ldots,x_{n-1})$, and $\mathcal{S}(x_1,\ldots,x_{n-1})$ are undetermined functions.

Some examples

In many physical cases we might generally expect that the two times, t and t', scale in the same way, i.e., $\phi_1=\phi_2\equiv\phi$. Below we explicitly list a few interesting examples in this category.

A simple situation corresponds to a scaling function, $\phi(t)$, which is asymptotically a power law in t (see references in [4]), i.e., one has $\phi(t)\sim t^{1/z}$, and the scaling of $C(t,t')$ is $C(t,t')=t^{f(t'/t)}\mathcal{S}(t'/t)$. In most cases we expect the exponent f to be a constant, so that

$$C(t,t')=t^{f/z}\mathcal{S}(t'/t). \tag{10}$$

Asymptotically, the ''power scaling'' of Eq. (10) is found in several toy models for glasses such as in a ''phase space'' model [16], the Backgammon entropic barriers model [18], the Queens long-range interactions model [19], in solvable models of interacting particles in high dimensionality [20], or in a kinetically constrained lattice gas [21] (see also references in [3]). Several of these cases are characterized by $f=0$. But in general one might expect cases with f different from 0. For instance, in the Bak-Sneppen SOC model, the two-time function $P(t,t')$, describing the return of activity to a site at time t that was most recently active at time t', for an avalanche started at $t=0$, seems to have a scaling of the form: $P(t,t')=t^{z_{BS}}\mathcal{P}(t'/t)$ [22], with constant z_{BS}. In a model of direct polymers in random media, similar behavior is found for the off-equilibrium ''overlap function'': $q(t,t')=t^{-x}\tilde{q}(t'/t)$ [23]. In some models of nonlinear diffusion equations [24], correlation functions have been shown to have a ''power law'' scaling structure of Eq. (10) with constant exponents f and with a scaling function $\mathcal{S}$ which is itself a power law. In the framework of our equilibrium dynamics, phenomena such as coarsening or, more generally, phase ordering kinetics in ''standard'' Ginzburg-Landau magnets usually show correlation functions which are asymptotically characterized (see [2]) by the above scaling of Eq. (10), which is often called *simple* or *full* or *naive aging*. In some discussions of glassy relaxation, a more complex, *interrupted aging*, scenario was also proposed [3,16], in which the long-time regime of the two-time autocorrelation function scales as $C(t,t')=\mathcal{S}(t'/t^{1+\mu})$. In the present picture this corresponds to two different power exponents for $\phi_1(t)\sim t^{1/z}$ and $\phi_2(t')\sim t'^{\,1/z+\mu}$.

In the case where $f=f(t'/t)$ is a nontrivial function of the scaling variable $\beta=t'/t$, one finds a *multiscaling* dynamical behavior. This is analogous to the multiscaling found, in a different context, by Coniglio and Zannetti in the spinodal decomposition of the $N=\infty$ Ginzburg Landau model with conserved order parameter or the one proposed also for the density profile of the DLA model [13].

In the previous cases, the scaling variable was the ratio of powers of the two involved times; however, in different situations, for instance in the limit in which the exponent $1/z$ goes to zero, one may expect to have a logarithmic behavior for ϕ: $\phi(t)\sim\ln(t)$. This situation gives as scaling structure $C(t,t')=\ln(t)^{f(\ln(t')/\ln(t))}\mathcal{S}(\ln(t')/\ln(t))$. In many cases one has $f=0$, namely,

$$C(t,t')=\mathcal{S}(\ln(t')/\ln(t)). \tag{11}$$

The *logarithmic scaling* of Eq. (11) is found in several systems. An example of diffusion that shows the logarithmic scaling is the one-dimensional Sinai model with a random local bias. In this case, for instance, the two-time residency probability asymptotically has a scaling form given by Eq. (11) with a scaling function $\mathcal{S}(\beta)$, which is an exponential corrected by a power law in $\beta=\ln(t')/\ln(t)$ [25]. Random field systems also show logarithmic scaling [2,3], but experimental random exchange Ising ferromagnets [26], among many others [2,3], also belong to this category. Logarithmic kinetics also have recently been experimentally observed in the amorphous-amorphous transformations in some glasses under high pressure [27]. Interestingly, nonthermal systems such as granular media, shaken at low vibration amplitudes, also present a nontrivial out- of-equilibrium dynamics, where numerical calculations on different models [17] suggest a logarithmic scaling in the relaxation of the two-time density correlation function as in Eq. (11).

The scenario for other disordered systems such as spin glass models is still controversial. To describe numerical calculations and to fit experimental data of relaxation in the thermoremanent magnetization of some spin glasses, several proposals, such as power scaling [Eq. (10)] and logarithmic scaling [Eq. (11)], have been made [28–31,3]. Also, in recent computer simulations of a Lennard-Jones off-equilibrium glass model the asymptotic behavior of the autocorrelation function was suggested to have a logarithmic scaling [32], as opposed to the power law scaling previously proposed [33].

III. MIXING CASE

Up to now we have dealt with *nonmixing* scale transformations, as in Eq. (2), where the scaling of each of the variables does not depend on the other. However, situations where *mixing* is present might be possible. Formally, the case of mixing may be dealt with as the nonmixing one; however, the results are too general to be of immediate practical use. For the sake of completeness, we just show them. In the mixing case one finds that Eq. (3) must be replaced by $C(t,t')=\phi_1(t,t')^{f(\beta)}\mathcal{S}(\beta)$, where the scaling variable β is now $\beta=\phi_2(t,t')/\phi_1(t,t')$. Here, as before, the $\phi_i(i=1,2)$ are two unknown functions fixed by the mixing transformations $\tilde{t}=F_1(t,t',l)$ and $\tilde{t}'=F_2(t,t',l)$ [with $F_1(t,t',1)=t$ and $F_2(t,t',1)=t'$]. The above result may be of little use because any function of two variables $C(t,t')$ may be written as above in terms of two other functions $\phi_1(t,t')$ and $\phi_2(t,t')$.

However, it may be interesting to work out a specific example of mixing transformations, that shows how one may recover, from simple scale principles, a multifractal scaling structure.

An example of mixing

While the function $C(t,t')$ has the general scaling property of Eq. (1), we now assume that the rescaling transformations of t and t' have the following specific form under a scale change of extension l:

$$\tilde{t}=t/l, \quad \tilde{t}'=t'/l^{z(t,t')}. \tag{12}$$

Interestingly, within this context, we find that the scaling of $C(t,t')$ is restricted to the following structure:

$$C(t,t')=t^{f(\beta)}\mathcal{S}(\beta). \tag{13}$$

Here the scaling variable β has only two possible forms: either it is a ratio of powers of the two times

$$\beta=t'/t^z \tag{14}$$

with $z=\text{const}$ (corresponding to a nonmixing case previously described), or

$$\beta=\frac{\ln(t')}{\ln(t)}+\frac{H(\beta)}{\ln(t)}, \tag{15}$$

where $H(\beta)$ is an undetermined function.

The scaling form (14) corresponds to the case $z(t,t')=\text{const}$, which is one of the nonmixing cases we dealt with before. The scaling form given in Eq. (15) corresponds instead to a nonconstant scaling exponent $z=z(t,t')$ in Eq. (2), which thus gives a mixing transformation of t and t'. Actually, it turns out that the only possible solution for a nonconstant z is $z(t,t')=\beta$, with β given in Eq. (15). In this case the scaling variable is asymptotically logarithmic in the two times, $\beta=\ln(t')/\ln(t)+O(1/\ln(t))$. This kind of scaling for different variables was proposed, for instance, for the multifractal description in turbulence [5,6], in the DLA model [7], in self-organized-critical (SOC) models [8] and in voltage distribution of random resistor networks [4,9]. These scaling forms, unlike ordinary critical phenomena, are characterized by a continuity of scaling exponents.

For definiteness it is interesting to work out the simple case where the function $H(\beta)$ is linear in β, a situation that might generically correspond to the case of very long times t and small β. By writing $H(\beta)=\beta\ln(t_0)-\ln(t_0')$ (where t_0 and t_0' are constants), from Eq. (15) one finds that $\beta=\ln(t'/t_0')/\ln(t/t_0)$. This case corresponds, for instance, to the multifractal scaling proposed by Kadanoff *et al.* in Ref. [8] to describe the avalanche size distribution in the context of SOC models.

Below, we work out the example of mixing transformation of Eq. (12) in detail. As in the general case above, we have to impose the group rules on the scale transformations. For the transformation of the variable t and t' this implies $(i=1,2)$

$$F_i(F_1(t,t',l_1),F_2(t,t',l_1),l_2)=F_i(t,t',l_1l_2). \tag{16}$$

For a transformation as in Eq. (12), this assertion simply imposes that

$$\frac{d}{dl}z(t(l),t'(l))=0. \tag{17}$$

The above Eq. (17) has two kinds of solution. The first is the trivial one: $z=\text{const}$. The second is nontrivial and has the following form:

$$z=\frac{\ln(t')}{\ln(t)}+\frac{H(z)}{\ln(t)}, \tag{18}$$

where $H(z)$ is a generic function. The latter may be obtained as follows. Since the function $z(\tilde{t},\tilde{t}')\equiv z(t/l,t'/l^z)$ is invariant under rescaling Eq. (17), we can write that $z(t/l,t'/l^z)=z(t,t')$. By fixing $l=t$, we obtain $z(t,t')=z(1,t'/t^z)$. Thus z is a function of the single variable t'/t^z, and we can write $z(t,t')=g(t'/t^z)$. Here we have defined

$$g(t'/t^z)\equiv z(1,t'/t^z). \tag{19}$$

By inverting the above relation, we have $t'/t^z=g^{-1}(z)$, and passing to the logarithms we recover $z=\ln(t')/\ln(t)+H(z)/\ln(t)$, where we have introduced the unknown generic function $H(z)=-\ln[g^{-1}(z)]$. Thus we have found the solution given in Eq. (18). This result states that, for fixed z, whenever $\ln(t')$ and $\ln(t)$ are large enough, we have $z=\ln(t')/\ln(t)+O[1/\ln(t)]$.

By then imposing group properties on the function $C(t,t')$ itself [see Eq. (7)], we obtain Eq. (8), which, after insertion of Eq. (12), implies that $f(t/l,t'/l^z)=f(t,t')$, and by taking $l=t$, as above, we have that $f(t,t')=f(1,t'/t^{z(t,t')})$.

Whenever z is a constant we recover a nonmixing case described in the preceding section. Let us now suppose that z is not a constant and is given by Eq. (18). In this case, we can prove that the exponent $f(t,t')$ is a function of the single variable z: $f(t,t')=f(z)$. In fact, as before we have that $f(t,t')=f(1,t'/t^z)=f(1,g^{-1}(z))$, i.e., f is a function of the variable z. Analogously one proves that $C(t,t')=t^{f(z)}\mathcal{S}(z)$, where the scaling function $\mathcal{S}$ is now $\mathcal{S}(x)=C(1,g^{-1}(x))$. From this result, in the asymptotic limit of large t' and t, we recover the scaling form for $C(t,t')$, given in Eqs. (13) and (15).

IV. CONCLUSIONS

We expect that the present approach may be useful to describe general properties of dynamical functions in physical systems when their characteristic times diverge, since, in such a situation, such as close to usual critical points, scale invariance should be reasonably present. Actually, one observes diverging characteristic times typically when out-of-equilibrium-dynamics phenomena become important, i.e., when an explicit dependence of functions such as $C(t,t')$ on both times (and not on their difference) is observed, a state which is sometimes generically called ''aging.'' In this perspective the structural properties of scaling described here should be naturally associated with out-of-equilibrium dynamics (i.e., with aging) effects. Interestingly, we have pointed out that in a broad variety of physical systems, ranging from magnets to polymers, glasses, or spin glasses, random fields, random ferromagnets, granular materials, diffusive systems, etc., one observes scaling properties of dynamical functions that may well be inserted in the framework reported above.

We have shown in full generality that a generalized homogeneous function, $C(t,t')$, which acts as in Eq. (1) under the scale transformation of its variables given in Eq. (2), must obey the scaling behavior of Eq. (3). In this theoretical framework, a multiscaling or multifractal behavior is also admissible in the dynamics. It would be interesting to determine if it exists in real dynamical systems.

The present approach is not restricted to scaling of dynamical functions. We have seen that it describes, as is well known, the usual scaling in standard critical phenomena, but it also describes multiscaling and multifractal properties introduced in apparently completely different systems such as, models of self-organized-criticality, DLA, random resistor networks. In this sense this approach may help to rationalize the existence of very few broad ''universality classes'' found in the scaling behaviors in very different contexts.

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